Girraween High School

2024

TRIAL HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 2

Total Marks: 100

Section 1 (Pages 2 – 5) **10 Marks**

- Attempt Q1 - Q10
- Allow about 15 minutes for this section

Section 2 (Pages 6-15) **90 marks**

- Attempt Q11 - Q16
- Allow about 2 hours and 45 minutes for this section

General Instructions

- Reading time: 10 minutes
- Working time: 3 Hours
- Write using a black or blue pen
- Board approved calculators may be used
- Laminated reference sheets are provided
- Answer multiple-choice questions by completely colouring in the appropriate circle on your multiple-choice answer sheet.
- Answer questions 11-16 in the appropriate answer booklet and show all relevant mathematical reasoning and/or calculations.

Section 1 (10 marks) Multiple choice

Attempt Questions 1-10

Allow about 15 minutes for this section

Question 1

The length of the vector $\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$ is

(A) 4 units (B) $\sqrt{22}$ units

(C) 8 units (D) $\sqrt{30}$ units

Question 2

The lines $\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 3 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 2 \\ -10 \end{pmatrix}$

(A) Are parallel. (B) Intersect with each other.

(C) Are skew. (D) Are the same line.

Question 3

When written using mathematical symbols, the statement "For all positive integers n there exists a real number x such that $x^3 = n$ is written

that $x^3 = n$"

(A) $\in n \in R \text{ such that } x^3 = n$ (B) $\forall n \in Z^+ \ni x \in R \text{ such that } x^3 = n$

(C) $\forall x \in z \ni n \in R \text{ such that } x^3 = n$ (D) $\in x \in R \text{ such that } x^3 = n$

Question 4

The contrapositive of the statement "If a number is real, its square will be real" is:

(A) If a number is NOT real its square won't be real.

(B) If the square of a number is real, the number will be real.

(C) If the square of a number is NOT real, the number is not real.

(D) If a number is real, its square root will be real.

Multiple choice continues on the following page

Multiple choice (continued)

Question 5

If $\overrightarrow{OA} = \underset{\sim}{a}, \overrightarrow{AB} = \underset{\sim}{b}$ and $\overrightarrow{BO} = \underset{\sim}{c}$, the converse of the statement

"If $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} = 0$ then O, A and B form a triangle is"

is:

(A) If O, A and B don't form a triangle then $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} \neq 0$

(B) If $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} \neq 0$ then O, A and B don't form a triangle

(C) If O, A and B form a triangle, then $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} \neq 0$

(D) If O, A and B form a triangle, then $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} = 0$.

Question 6

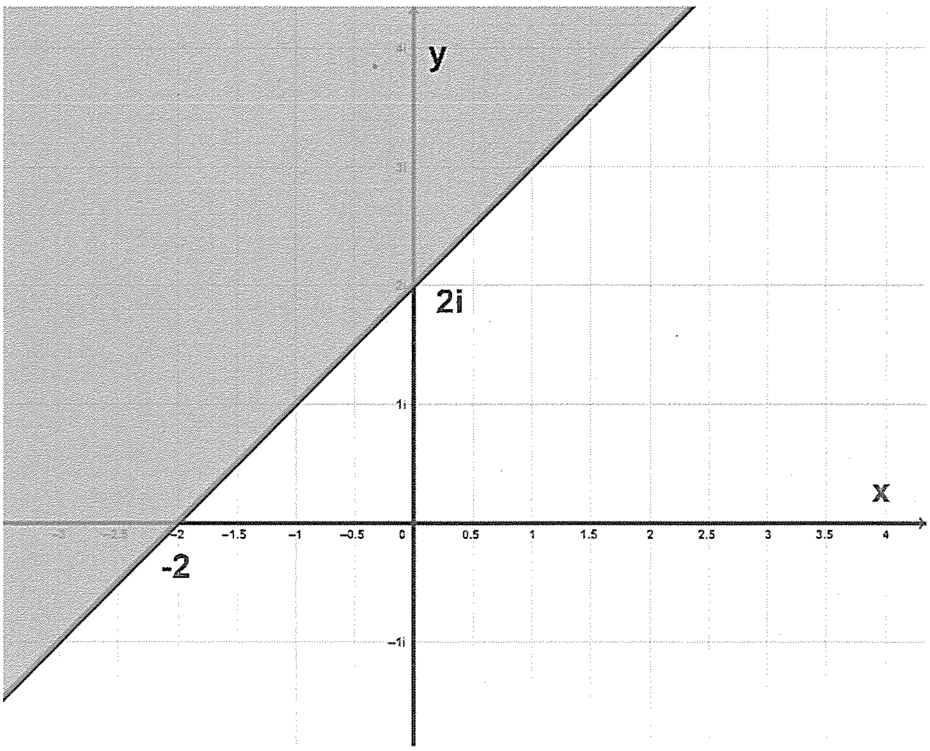
The shaded area in the Argand diagram below is best represented by

(A) $|z + 3 + i| \geq |z + 1 - 5i|$

(B) $|z - 3 - i| \geq |z + 1 - 5i|$

(C) $|z + 1 - 5i| \geq |z + 3 + i|$

(D) $|z + 1 - 5i| \geq |z - 3 - i|$



Multiple choice continues on the following page

Question 7

$$\int x \sin x . dx =$$

(A) $\sin x - x \cos x + C$

(B) $\cos x - x \sin x + C$

(C) $\sin x + x \cos x + C$

(D) $\cos x + x \sin x + C$

Question 8

Which of the following integrations does NOT require a tan^{-1} function when integrated?

(A) $\displaystyle\int \frac{x-3}{(x^2+2x+2)(x-3)} . dx$

(B) $\displaystyle\int \frac{x}{(x^2+2x+2)(x-3)} . dx$

(C) $\displaystyle\int \frac{x^2-2x-3}{(x^2+2x+2)(x-3)} . dx$

(D) $\displaystyle\int \frac{x^2-3x}{(x^2+2x+2)(x-3)} . dx$

Question 9

A particle moving with simple harmonic motion (SHM) is oscillating between x = 1 and x = 5. If it starts at x = 3 and first returns to x = 3 after 4 seconds its displacement equation with respect to time could be

(A) $x = 2\cos \frac{1}{4}t + 3$

(B) $x = 2\sin \frac{1}{4}t + 3$

(C) $x = 2\cos \frac{\pi}{4}t + 3$

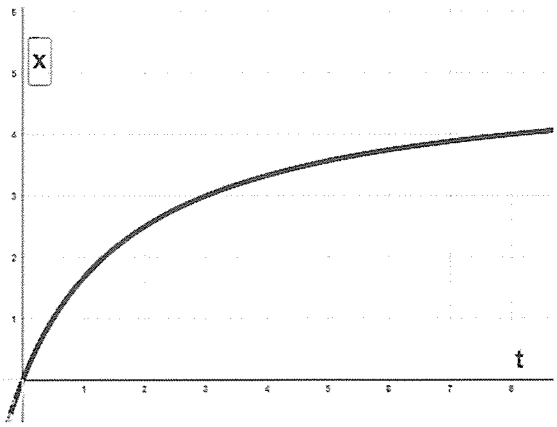
(D) $x = 2\sin \frac{\pi}{4}t + 3$

Multiple choice continues on the following page

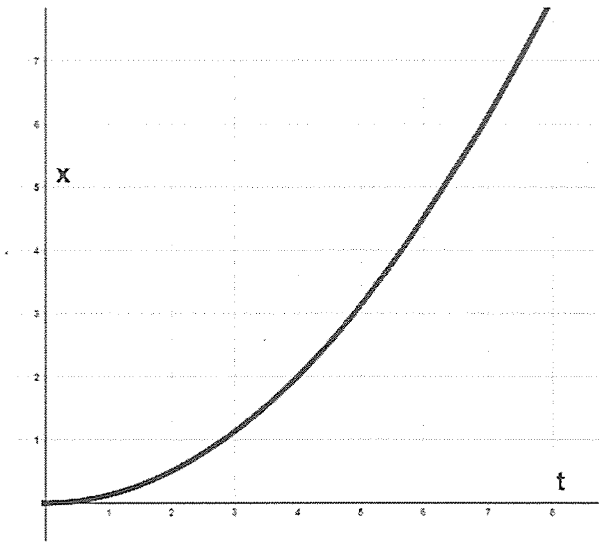
Question 10

Which of the following graphs represents displacement (x) against time (t) if if $v > 0$ and $a < 0$?

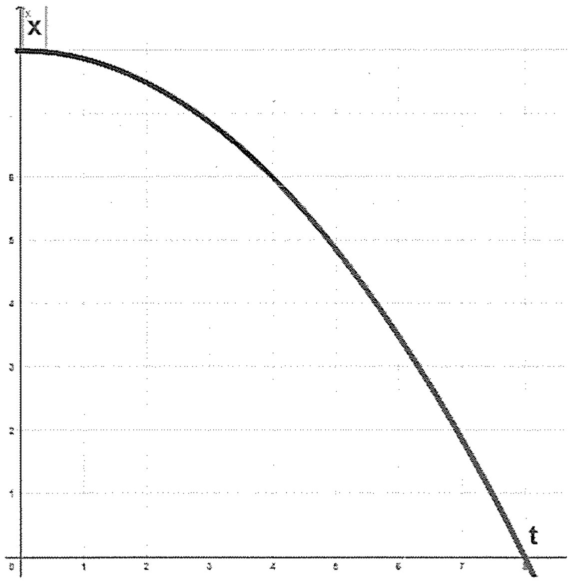
(A)



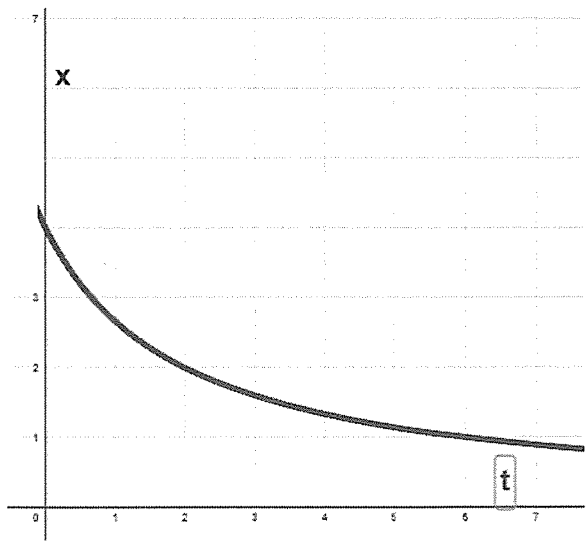
(B)



(C)



(D)



Examination continues on the following page

Section II (90 marks)

Attempt Questions 11-16

Allow about 2 hours and 45 minutes for this section

Start the answers to each question on a separate page in your answer booklet.

In Questions 11-16 your responses should include all relevant mathematical reasoning and/ or calculations. *(Note: for complex number questions, the notation cis is acceptable).*

Question 11 ***(13 marks)*** **Marks**

(a) **(i)** Show that $\frac{1+3i}{2+i} = 1+i$ **1**

(ii) State $1+i$ in modulus/argument form. **1**

(iii) Hence show that $tan^{-1}(3) - tan^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{4}$ **1**

(b) **(i)** By letting $\sqrt{15+8i} = x+iy,\ x,\ y\ real$, show that **3**

$\sqrt{15+8i} = \pm(4+i)$

(ii) Hence solve $z^2 + (2-3i)z - 5(1+i) = 0$. **2**

(c) Sketch and shade the region in the complex plane where **3**

$|z-2| \leq 2$ and $-\frac{\pi}{2} \leq Arg\ z \leq -\frac{\pi}{4}$ hold simultaneously.

(d) Use DeMoivre's Theorem to find all of the 5th roots of $16 - 16i\sqrt{3}$. **2**

Leave your answers in modulus/argument form.

Examination continues on the following page

Question 12 ***(14 marks)*** **Marks**

(a) **(i)** Express $\dfrac{5x^2 - 14x + 17}{(x^2+1)(x-3)}$ in the form $\dfrac{Ax+B}{x^2+1} + \dfrac{C}{x-3}$. 3

(ii) Hence find $\displaystyle\int \frac{5x^2 - 14x + 17}{(x^2+1)(x-3)}\,.dx$ 1

(b) Find $\displaystyle\int e^x \cos x\,.dx$ 3

(c) Find $\displaystyle\int \sqrt{\frac{x}{4-x}}\,.dx$ 3

(d) If $I_n = \displaystyle\int_0^{\frac{1}{\sqrt{2}}} \left(\sqrt{1-x^2}\right)^n.dx$,

(i) Show that $I_n = J_{n+1}$ where $J_{n+1} = \displaystyle\int_0^{\frac{\pi}{4}} \cos^{n+1}\theta.d\theta$ 1

(ii) Hence find I_3. 3

Examination continues on the following page

Question 13 ***(14 marks)*** **Marks**

(a) If a particle is moving so that its position x at time t is given by $x = 2\sqrt{3}\, sin\, 4t + 6\, cos\, 4t$:

(i) Show that the particle is moving in Simple Harmonic Motion **1**

(ii) By expressing the particle's motion in the form $x = A\, sin\, (4t + \alpha)$, find the period and amplitude of the motion. **2**

(b) Another particle is moving so that $4v^2 = 8 - x^2 - 2x$. Show that the particle is moving with simple harmonic motion and find the centre, period and amplitude of the motion. **4**

(c) **(i)** Prove by contradiction that $\sqrt{7}$ is irrational. **3**

(ii) Prove by contraposition that the square root of an irrational number is also irrational. **2**

(iii) Hence prove that $\sqrt{2} + \sqrt{7}$ is irrational (you may assume that $\sqrt{14}$ is irrational and that the sum of a rational number and an irrational number is irrational). **2**

Examination continues on the following page

Question 14 *(19 marks)* **Marks**

(a) (i) Solve $k^2 - k - 2 \geq 0$ **2**

(ii) Hence prove by induction that $3^n > 2n^2 - 3 \; \forall \, n \in Z^+ \geq 2$ **3**

(b) (i) Prove $a^2 + b^2 \geq 2ab \; \forall \; a, b \in R$ **1**

(ii) Hence prove $a^2 + b^2 + c^2 \geq ab + ac + bc \; \forall a, b, c \in R$ **2**

(iii) Hence prove if $a^2 + b^2 + c^2 = 27, -9 \leq a + b + c \leq 9$ **2**

(iv) State the centre and radius of the sphere **1**

$(x - 1)^2 + (y + 2)^2 + (z + 3)^2 = 27$

(v) Hence show that if $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ is a point on the surface of the sphere **1**

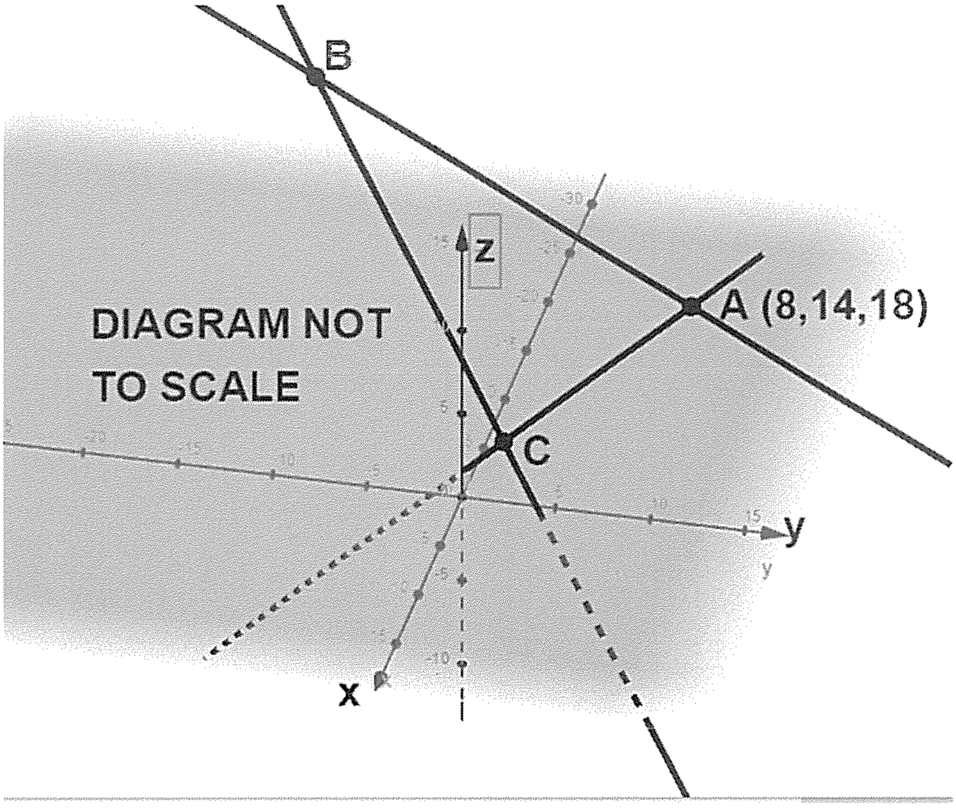
$(x - 1)^2 + (y + 2)^2 + (z + 3)^2 = 27$, then

$-13 \leq x_1 + y_1 + z_1 \leq 5$

Question 14 continues on the following page

Question 14 (continued) **Marks**

(c) $A\begin{pmatrix}8\\14\\18\end{pmatrix}$, *B and C* are points in three-dimensional space. *(See diagram.)*



(i) The line AB has equation $\begin{pmatrix}8\\14\\18\end{pmatrix}+\lambda\begin{pmatrix}3\\4\\0\end{pmatrix}$ and the line BC has equation **2**

$\begin{pmatrix}-16\\-18\\28\end{pmatrix}+\mu\begin{pmatrix}3\\4\\-5\end{pmatrix}$. Show *by solving the equations for AB and BC simultaneously* that B is the point $\begin{pmatrix}-10\\-10\\18\end{pmatrix}$.

(ii) Find $\angle ABC$. **1**

(iii) AC is the line $\begin{pmatrix}8\\14\\18\end{pmatrix}+\lambda\begin{pmatrix}-3\\-4\\-5\end{pmatrix}$. Show AC is perpendicular to BC. **2**

Hence show that $\Delta\, ABC$ is isosceles.

(iv) Find the area of $\Delta\, ABC$ **2**

Examination continues on the following page

Question 15 ***(15 marks)*** **Marks**

(a) A 100g ball is thrown horizontally into a headwind at 20m/s. It experiences resistance from the wind of 0.5 Newtons, as well as air resistance of 0.05v Newtons, where v is the current velocity of the ball. Ignoring gravity(!!!)

(i) Show that the acceleration of the ball at time t is given by $\ddot{x} = -5 - 0.5v$. **1**

(ii) Show that the time taken from when the ball is initially thrown to when it reaches a certain velocity is given by **3**

$t = -2ln\left(\frac{10+v}{30}\right)$

and find the time at which the ball stops moving forwards.

(iii) Show that the ball's position at time t is given by $x = 60 - 60e^{-0.5t} - 10t$ and find how far the ball has travelled before it stops moving forwards. **2**

(b) A relief parcel is dropped vertically from a stationary helicopter 50m above the ground. It experiences acceleration due to gravity of $10m/s^2$ and acceleration due to air resistance against its direction of motion of $\frac{v^2}{40}\ m/s^2$.

(i) Taking down as positive, express acceleration in terms of velocity and find the parcel's terminal velocity. **1**

(ii) Show that the distance the parcel has dropped from the helicopter is given by $x = -20\ ln\left(\frac{400 - v^2}{400}\right)$ and find the speed at which the parcel hits the ground. **4**

Question 15 continues on the following page

Question 15 (continued) **Marks**

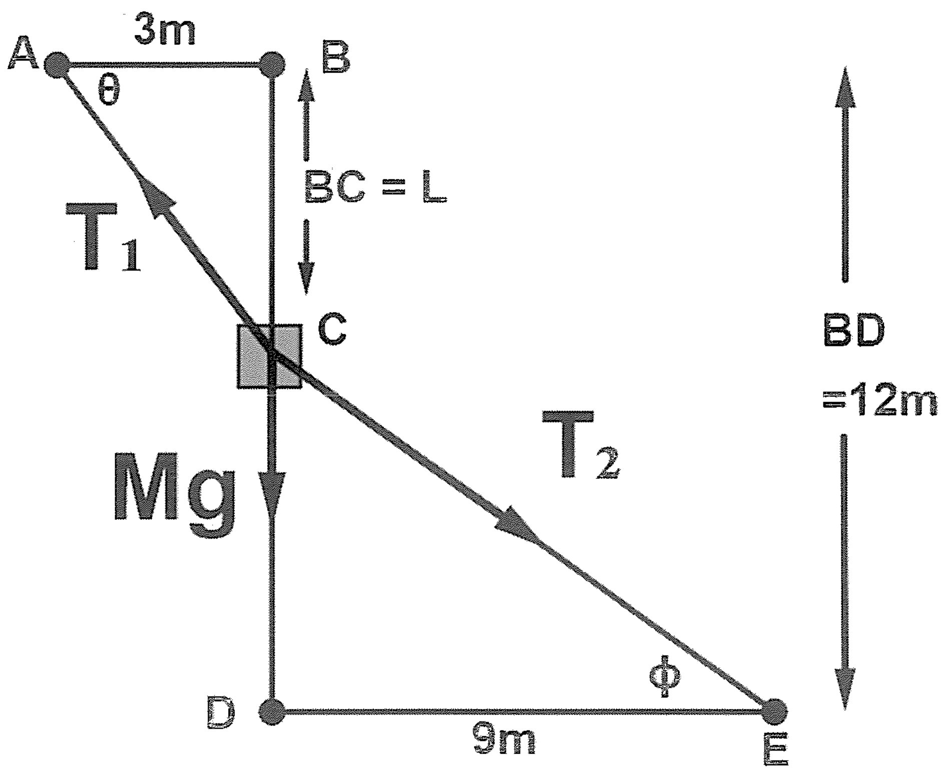
(c) A machine with mass M kg is being raised vertically at a constant rate. It is at C, which is L metres below the roof of a room with total height from D to B of 12 metres. It is attached to two ropes:

one at A (on the roof, 3 metres to the left of the machine with a tension of T_1 Newtons) and

one at B (on the floor, 9 metres to the right of the machine with a tension of T_2 Newtons)

The two ropes make angles of θ and ϕ respectively with the horizontal.

(see diagram)



(i) By resolving forces vertically and horizontally, show that 3

$$\tan\theta = \tan\phi + \frac{Mg}{T_2\cos\phi}$$

(ii) Show that the machine can't be lifted 9 metres or more above the floor. 1

Question 16 *(15 marks)* **Marks**

(a) If $z = e^{\frac{3\pi i}{4}}$ and $w = e^{\frac{2\pi i}{3}}$,

(i) Show that $|z + w|^2 = \frac{4+\sqrt{2}+\sqrt{6}}{2}$. **2**

(ii) If $\overrightarrow{OA} = z, \overrightarrow{OB} = w \text{ and } \overrightarrow{OC} = z + w$, **1**

show that $\angle COB = \frac{\pi}{24}$.

(iii) Show that $\frac{z+w}{w} = \frac{4+\sqrt{2}+\sqrt{6}}{4} + i\left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)$. **2**

(iv) Hence show that $\cos\frac{\pi}{24} = \frac{\sqrt{2}\sqrt{4+\sqrt{2}+\sqrt{6}}}{4}$ **1**

Question 16 continues on the following page

Question 16 (continued) | Marks

(b) (i) By plotting points for $x = -4, -3, -2, -1, 0, 1, 2, 3, and\ 4$, 1

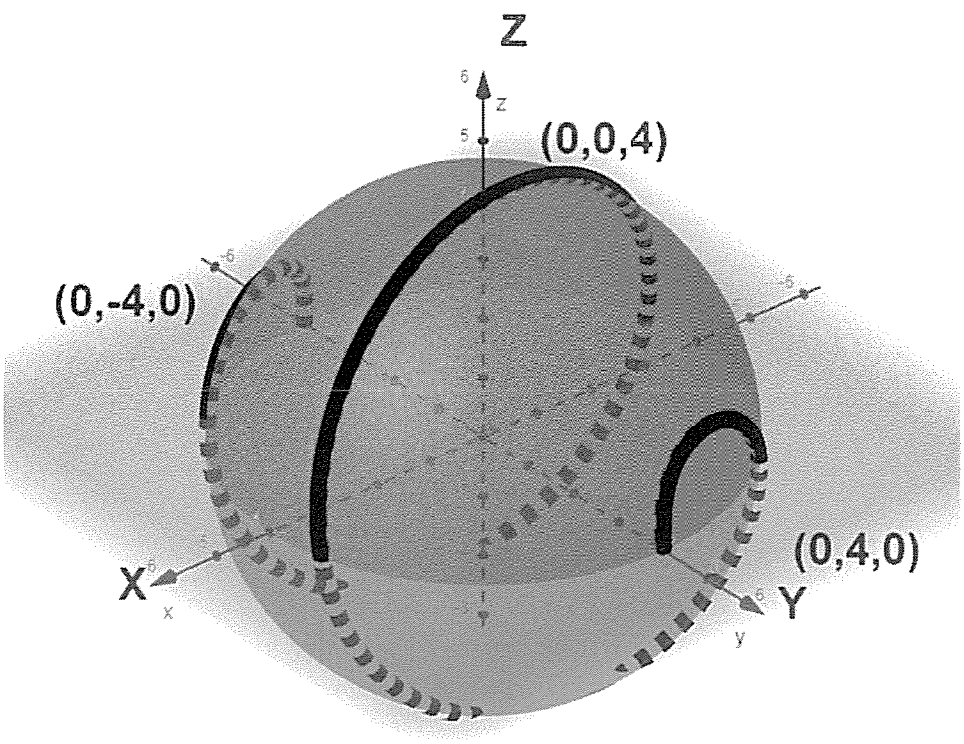
sketch $y = \sqrt{16 - x^2}\, sin\left(\frac{\pi}{2}x\right)$.

(ii) The spiral below is on the surface of the sphere 3

$x^2 + y^2 + z^2 = 16$. It passes through the points (0,-4,0),

(0,0,4) and (0,4,0). It also passes through the points $\sqrt{15}, 1, 0)$

$(0, 2, -\sqrt{12})\ and\ (-\sqrt{7}, 3, 0)$. Find the parametric equation of the spiral.



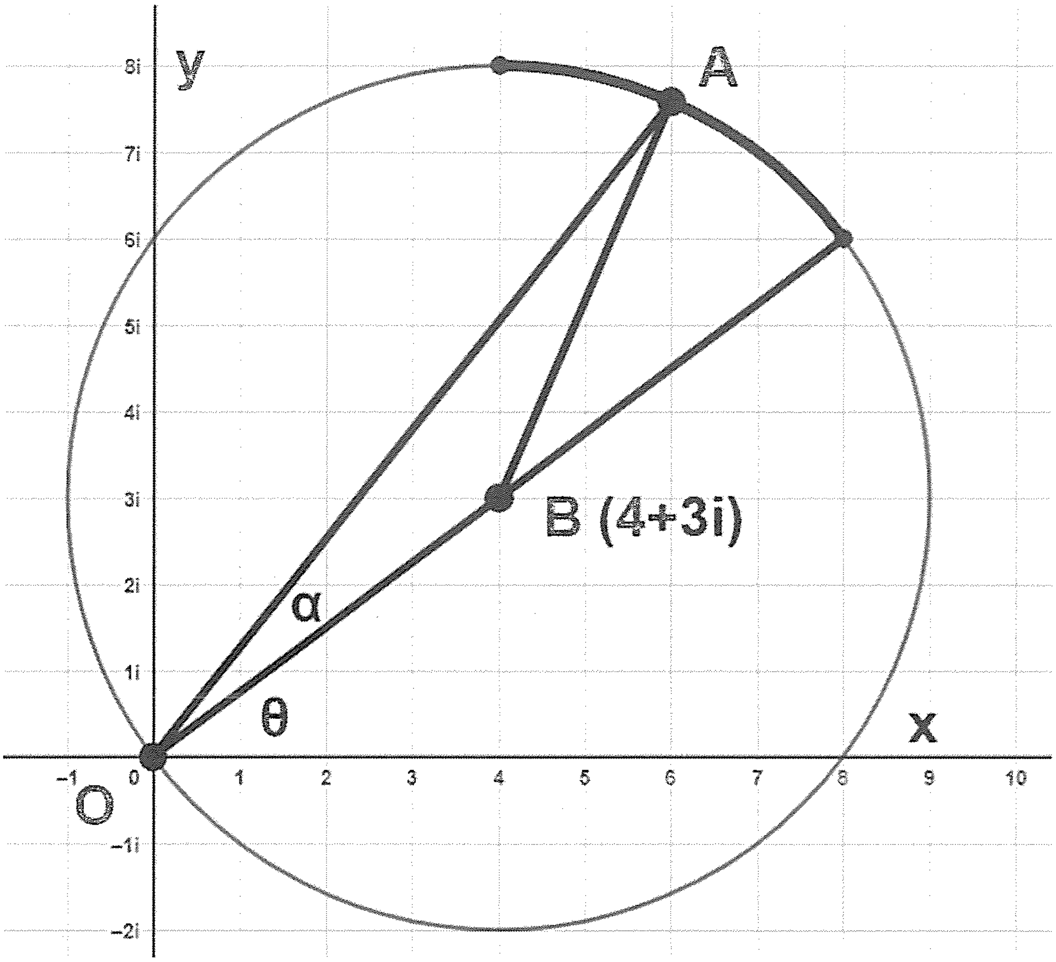
Question 16 continues on the following page

(c) The complex number z lies in the complex plane so that

$|z - 4 - 3i| = 5$ and

$tan^{-1}\left(\frac{3}{4}\right) \leq Arg\,(z - 4 - 3i) \leq \left(\frac{\pi}{2}\right)$.

If $\overrightarrow{OA} = z, \overrightarrow{OB} = 4 + 3i$ and $\angle BOX = \theta$ and $\angle AOB = \alpha$ *(see diagram)*



(i) Show that $|z| = 10\,cos\,\alpha$. **2**

(ii) $If\ w = \frac{1}{z}$, show that **2**

$$w = \frac{1}{10\,cos\,\alpha}\left\{\left(\frac{4}{5}\,cos\,\alpha - \frac{3}{5}\,sin\,\alpha\right) - i\left(\frac{3}{5}\,cos\,\alpha + \frac{4}{5}\,sin\,\alpha\right)\right\}$$

(iii) Hence or otherwise find the Cartesian equation for the locus of w. **1**

(Note: you do NOT need to find the domain or range of w).

END OF EXAMINATION

GIRRAWEEN HIGH SCHOOL

MATHEMATICS EXTENSION 2
2024 TRIAL HIGHER SCHOOL CERTIFICATE

FINAL

Student Number: Solutions – Marking

This Booklet contains the answer sheet for Section 1 and Writing Booklet for Section 2.

Section 1 ANSWER SHEET

Select the alternative A, B, C or D that best answers the question.

1.	A	○	B	○	C	○	D	●
2.	A	○	B	○	C	○	D	●
3.	A	○	B	●	C	○	D	○
4.	A	○	B	○	C	●	D	○
5.	A	○	B	○	C	○	D	●
6.	A	○	B	●	C	○	D	○
7.	A	●	B	○	C	○	D	○
8.	A	○	B	○	C	●	D	○
9.	A	○	B	○	C	○	D	●
10.	A	●	B	○	C	○	D	○

Instructions

- If you need more paper for Section 2, please ask your supervisor.
- Write your student number on every booklet you use.
- Write on both sides of each sheet of paper.

Total number of booklets used ______________.

Solutions p-1

Solutions p.2

MC Solutions

(1) Length $= \sqrt{1^2+(-2)^2+5^2}$

$= \sqrt{30}$ units (D)

(2) $\begin{pmatrix}-4\\2\\-10\end{pmatrix} = -2\begin{pmatrix}2\\-1\\5\end{pmatrix}$ so lines are either $\parallel$ or same line.

Solving

$\begin{pmatrix}3\\4\\1\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\5\end{pmatrix} = \begin{pmatrix}5\\3\\6\end{pmatrix}$

$3+2\lambda=5$
$4-\lambda=3 \Rightarrow \lambda=1$ in all cases.
$1+5\lambda=6$

SAME LINE (D)

(3) $\forall n \in Z^+ \ \exists x \in R$ such that $x^3=n$ (B)

(4) If the square of a number isn't real, the number isn't real (C)

(5) If O, A & B form a triangle then $\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}=0$ (D)

(6) Shaded area is $y \geq x+2$.

Simplifying (B)

$|z-3-i| \geq |z+1-5i|$

$(x-3)^2+(y-1)^2 \geq (x+1)^2+(y-5)^2$

which simplifies to

$y \geq x+2$ (B)

Alternatively, its all of the parts of the complex plane that are closer to $-1+5i$ than $3+i$.

(7) $\int x\sin x\,.dx$

$u=x \quad v=-\cos x$

$u'=1 \quad v'=\sin x$

By

$\int uv'.dx = uv - \int vu'.dx$

$\int x\sin x\,.dx$

$= -x\cos x + \int \cos x\,.dx$

$= \sin x - x\cos x + C$ (A)

(8) (A) simplifies to

$\int \frac{1}{(x+1)^2+1}.dx$ which needs a $\tan^{-1}$.

(B) simplifies to

$\int \frac{x-3}{(x^2+2x+2)(x-3)} + \frac{3}{(x^2+2x+2)(x-3)}.dx$

$= \int \frac{1}{x^2+2x+2} + \frac{3}{(x^2+2x+2)(x-3)}.dx$

which needs a $\tan^{-1}$.

(C) Simplifies to

$\int \frac{x+1}{x^2+2x+2}.dx$ (C)

which integrates to

$\frac{1}{2}\ln(x^2+2x+2)+C$

(d) simplifies to

$\int \frac{x}{x^2+2x+2}.dx = \int \frac{x+1}{x^2+2x+2} - \frac{1}{x^2+2x+2}.dx$

which needs a $\tan^{-1}$.

(9) Starts at centre $\rightarrow$ sin

Period $= 8$, $\frac{2\pi}{n}=8 \Rightarrow n=\frac{\pi}{4}$.

Amplitude $= 2$.

Centre $= 3$.

$x = 2\sin\frac{\pi}{4}t+3$ (D)

(10) x increasing at a decreasing rate $\Rightarrow$ (A)

Solutions: p. 3

Q.(11)(a)(i) $\dfrac{1+3i}{2+i} \times \dfrac{(2-i)}{(2-i)}$

$= \dfrac{5+5i}{5}$

$= \underline{1+i}$

(ii) $1+i = \sqrt{2}\,\text{cis}\,\dfrac{\pi}{4}$.

(iii) Hence as $1+3i = \sqrt{10}\,\text{cis}\left(\tan^{-1}(3)\right)$

& $2+i = \sqrt{5}\,\text{cis}\left(\tan^{-1}\left(\tfrac{1}{2}\right)\right)$

$\dfrac{1+3i}{2+i} = \sqrt{2}\,\text{cis}\left[\tan^{-1}(3) - \tan^{-1}\left(\tfrac{1}{2}\right)\right] = 1+i = \sqrt{2}\,\text{cis}\,\dfrac{\pi}{4},$

$\underline{\tan^{-1}(3) - \tan^{-1}\left(\tfrac{1}{2}\right) = \dfrac{\pi}{4}}$.

(b)(i) $(x+iy)^2 = 15+8i$.

$(x^2-y^2)+2ixy = 15+8i$.

Equating reals, Equating imaginaries

$x^2-y^2 = 15 \;(1)$ $\quad 2xy = 8$

$y = \dfrac{4}{x} \;(2)$

Sub. (2) in (1): $x^2 - \dfrac{16}{x^2} = 15$

$x^4 - 15x^2 - 16 = 0$

$(x^2-16)(x^2+1) = 0$

As x is real, $x = \pm 4$ only

As $y = \dfrac{4}{x}$, $y = \pm 1$.

$\underline{\therefore \sqrt{15+8i} = \pm(4+i)}$.

(ii) Solving $z^2+(2-3i)z-5(1+i) = 0$

$z = \dfrac{3i-2 \pm \sqrt{(2-3i)^2 - 4\times 1\times -5(1+i)}}{2\times 1}$

$= \dfrac{3i-2 \pm \sqrt{15+8i}}{2}$

$= \dfrac{3i-2 \pm (4+i)}{2}$ [from (i)]

$\underline{z = 1+2i \text{ or } z = -3+i}$.

Marker's Comments:

Generally very well done.

(ii) Well done.

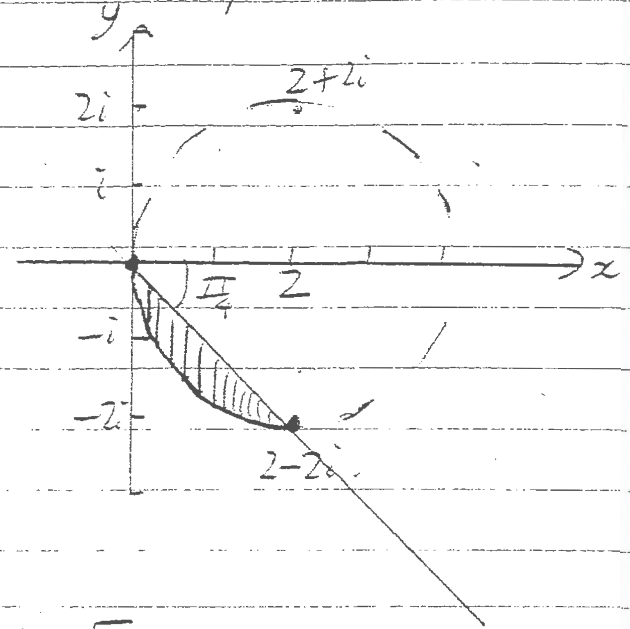
(iii) Well done.

(b)(i) Generally well done BUT some students used
$x^2+y^2 = 17$
$x^2-y^2 = 15$.
You CAN do this BUT there IS a danger.
If you use $x^2+y^2 = 17$
$y = \dfrac{4}{x}$
you will get more possibilities than actually exist so BE CAREFUL.

Best to avoid $x^2+y^2=17$ altogether.

Generally well done.

Q.(11)(c)



Generally well understood but SLOPPILY DRAWN.
Students should ALWAYS
→ mark in circle centre.
→ mark in point of intersection of circle & line.
→ Draw the part of the circle which is OUTSIDE the area DOTTED.
→ Use a RULER for lines & axes.

(d) $16-16i\sqrt{3}$

$= 32\operatorname{cis}\left(\frac{5\pi}{3}\right)$ (or $32\operatorname{cis}\left(\frac{-\pi}{3}\right)$)

By De Moivre, 5^{TH} roots $= 2\operatorname{cis}\frac{\pi}{3}+\frac{2k\pi}{5}$, $k=0,1,2,3,4$

i.e. $z = 2\operatorname{cis}\frac{\pi}{3}, 2\operatorname{cis}\frac{11\pi}{15}, 2\operatorname{cis}\frac{17\pi}{15}, 2\operatorname{cis}\frac{23\pi}{15}, 2\operatorname{cis}\frac{29\pi}{15}$

or using principal arguments
$z = 2\operatorname{cis}\left(-\frac{13\pi}{15}\right), 2\operatorname{cis}\left(-\frac{7\pi}{15}\right), 2\operatorname{cis}\left(\frac{-\pi}{15}\right), 2\operatorname{cis}\left(\frac{\pi}{3}\right), 2\operatorname{cis}\left(\frac{11\pi}{15}\right)$

Generally very well done.
A small minority of weaker students had trouble with this.

Solutions: p.5

Q.(12)(a)(i) $\dfrac{5x^2-14x+17}{(x^2+1)(x-3)} \equiv \dfrac{Ax+B}{x^2+1} + \dfrac{C}{x-3}$

$\times (x^2+1)(x-3)$

$5x^2-14x+17 = (Ax+B)(x-3) + C(x^2+1) \quad (1)$

Sub. $x=3$ in (1)

$20 = C\times 10$

$2 = C$

Sub. $x=0$, $C=2$ in (1):

$17 = -3B + 2\times 1$

$-5 = B$

Sub. $x=1$, $B=-5$, $C=2$ in (1):

$8 = -2(A-5)+4$

$3 = A$

$\therefore A=3, B=-5, C=2$

$\therefore \dfrac{5x^2-14x+17}{(x^2+1)(x-3)} = \dfrac{3x-5}{x^2+1} + \dfrac{2}{x-3}$

Marker's Comments

(ii) Hence $\displaystyle\int \frac{5x^2-14x+17}{(x^2+1)(x-3)}\,dx$

$$= \int \frac{3x-5}{x^2+1} + \frac{2}{x-3}\,dx$$

$$= \int \frac{3x}{x^2+1}\,dx - \int \frac{5}{x^2+1}\,dx + \int \frac{2}{x-3}\,dx$$

$$= \frac{3}{2}\int \frac{2x}{x^2+1}\,dx - \int \frac{5}{x^2+1}\,dx + \int \frac{2}{(x-3)}\,dx$$

$$= \frac{3}{2}\ln[x^2+1] - 5\tan^{-1}(x) + 2\ln(x-3) + C.$$

Most students have done this Question very well and very few students made calculation mistakes.

Q.(12)(b) $\int e^x \cos x . dx$ $\quad u = e^x \quad v = \sin x$
$\quad u' = e^x \quad v' = \cos x$

By $\int uv'.dx = uv - \int vu'.dx$

$\int e^x \cos x . dx = e^x \sin x - \int e^x \sin x . dx \quad (1)$

Taking $\int e^x \sin x . dx$ out of (1): $\quad u = e^x \quad v = -\cos x$
$\quad u' = e^x \quad v' = \sin x$

By $\int uv'.dx = uv - \int vu'.dx$

$\int e^x \sin x . dx = -e^x \cos x + \int e^x \cos x . dx \quad (2)$

Sub. (2) in (1):

$\int e^x \cos x . dx = e^x \sin x + e^x \cos x - \int e^x \cos x . dx$

$2\int e^x \cos x . dx = e^x(\sin x + \cos x)$

$\int e^x \cos x . dx = \frac{e^x(\sin x + \cos x)}{2} + C.$

Note: You can also stat using $u = \cos x \quad v = e^x$
$u' = -\sin x \quad v' = e^x$

& will get the same result.

Marker's Commets:

done Very well. Almost all students got full marks for this Question

Solutions: p. 7

Marker's Comment

$$Q.(12)(c) \int \sqrt{\frac{x}{4-x}}\,.dx$$

EITHER

$x = 4\sin^2\theta \quad \therefore \quad dx = 8\sin\theta\cos\theta\,.d\theta$

$4 - x = 4\cos^2\theta$

$$= \int \sqrt{\frac{4\sin^2\theta}{4\cos^2\theta}}\,.8\sin\theta\cos\theta\,.d\theta$$

$$= \int \frac{\sin\theta}{\cos\theta}\,.8\sin\theta\cos\theta\,.d\theta$$

$$= \int 8\sin^2\theta\,.d\theta$$

$$= \int 4 - 4\cos 2\theta\,.d\theta$$

$$= 4\theta - 2\sin 2\theta + C$$

$$= 4\theta - 4\sin\theta\cos\theta + C.$$

done very well but students lost marks for silly mistakes

As $x = 4\sin^2\theta$, $\sqrt{x} = 2\sin\theta$, $\sqrt{4-x} = 2\cos\theta$ & $2\sin\theta \times 2\cos\theta = 4\sin\theta\cos\theta$

$$\underline{\int \sqrt{\frac{x}{4-x}}\,.dx = 4\sin^{-1}\left(\frac{\sqrt{x}}{2}\right) - \sqrt{x(4-x)} + C.}$$

Marker's Comments

OR Rationalising the numerator:

$$\int \sqrt{\frac{x}{4-x}}\,.dx$$

$$= \int \frac{x}{\sqrt{4x - x^2}}\,.dx$$

$$= \int \frac{x-2}{\sqrt{4x-x^2}}\,.dx + \int \frac{2}{\sqrt{4-(x-2)^2}}\,.dx$$

$$= -\frac{1}{2}\int \frac{4-2x}{\sqrt{4x-x^2}}\,.dx + \int \frac{2}{\sqrt{4-(x-2)^2}}\,.dx$$

$$= -\sqrt{x(4-x)} + 2\sin^{-1}\left(\frac{x-2}{2}\right) + C.$$

$$\therefore \underline{\int \sqrt{\frac{x}{4-x}}\,.dx = 2\sin^{-1}\left(\frac{x-2}{2}\right) - \sqrt{x(4-x)} + C.}$$

These two answers ARE equal, as

$$\sin^{-1}\left(\frac{x-2}{2}\right) = 2\sin^{-1}\frac{\sqrt{x}}{2} - \frac{\pi}{2}.$$

[& $\frac{\pi}{2}$ would be in $+C$].

Showing: Let $\frac{\sqrt{x}}{2} = \sin\theta$.

$\sin\left(2\theta - \frac{\pi}{2}\right) = \sin 2\theta\cos\frac{\pi}{2} - \cos 2\theta \sin\frac{\pi}{2}$

$= -\cos 2\theta$

$= 2\sin^2\theta - 1$

$= 2 \times \frac{x}{4} - 1 = \frac{x-2}{2}.$

Solutions: p.8

Q.(12)(d)(i) $I_n = \int_0^{\frac{1}{\sqrt{2}}} \left(\sqrt{1-x^2}\right)^n . dx$

Let $x = \sin\theta$

$dx = \cos\theta . d\theta$

Marker's Comments

$= \int_0^{\frac{\pi}{4}} (\cos\theta)^n . \cos\theta . d\theta$

done very well almost all students used double angles to solve ii

$= \int_0^{\frac{\pi}{4}} \cos^{n+1}\theta . d\theta.$

(ii) $I_3 = J_4 = \int_0^{\frac{\pi}{4}} \cos^4\theta . d\theta.$

Finding J_{n+1}:

$\int_0^{\frac{\pi}{4}} \cos^{n+1}\theta . d\theta$ $\quad u = \cos^n\theta \quad v = \sin\theta$

$u' = n\cos^{n-1}\theta . -\sin\theta \quad v' = \cos\theta$

By $\int uv' . d\theta = uv - \int v u' . d\theta$

$\int \cos^{n+1}\theta . d\theta = \left[\cos^n\theta \sin\theta\right]_0^{\frac{\pi}{4}} + n\int_0^{\frac{\pi}{4}} \cos^{n-1}\theta \sin^2\theta . d\theta$

$= \left[\left(\frac{\sqrt{2}}{2}\right)^n \times \left(\frac{\sqrt{2}}{2}\right)\right] + n\int_0^{\frac{\pi}{4}} \cos^{n-1}\theta . d\theta - n\int_0^{\frac{\pi}{4}} \cos^{n+1}\theta . d\theta$

[using $\sin^2\theta = 1 - \cos^2\theta$].

$J_{n+1} = \left(\frac{\sqrt{2}}{2}\right)^{n+1} + nJ_{n-1} - nJ_{n+1}$

Marker's comments

$(n+1)J_{n+1} = \left(\frac{\sqrt{2}}{2}\right)^{n+1} + nJ_{n-1}$

$J_{n+1} = \frac{\left(\frac{\sqrt{2}}{2}\right)^{n+1}}{n+1} + \frac{n}{n+1} J_{n-1}.$

Building down

$J_4 = \frac{\left(\frac{\sqrt{2}}{2}\right)^4}{4} + \frac{3}{4} J_2$

$= \frac{1}{16} + \frac{3}{4} J_2$

$J_2 = \frac{\left(\frac{\sqrt{2}}{2}\right)^2}{2} + \frac{1}{2} J_0$

$= \frac{1}{4} + \frac{1}{2} J_0$

$J_0 = \int_0^{\frac{\pi}{4}} 1 . dx = \frac{\pi}{4}.$

Building up:

$J_2 = \frac{1}{4} + \frac{1}{2} \times \frac{\pi}{4}$

$= \frac{1}{4} + \frac{\pi}{8}$

$J_4 = \frac{1}{16} + \frac{3}{4}\left(\frac{1}{4} + \frac{\pi}{8}\right)$

$= \frac{1}{4} + \frac{3\pi}{32}.$

$\therefore I_3 = \int_0^{\frac{1}{\sqrt{2}}} \sqrt{1-x^2} . dx = \frac{1}{4} + \frac{3\pi}{32}.$

Solutions: p. 9

Q.(13)(a) $x = 2\sqrt{3}\sin 4t + 6\cos 4t$.

(i) $\dot{x} = 8\sqrt{3}\cos 4t - 24\sin 4t$

$\ddot{x} = -32\sqrt{3}\sin 4t - 96\cos 4t$

$\ddot{x} = -16(2\sqrt{3}\sin 4t + 6\cos 4t)$

$\therefore \ddot{x} = -16x$

$\rightarrow$ Moving with SHM as $\ddot{x} = -n^2 x$, $n = 4$.

(ii) $2\sqrt{3}\sin 4t + 6\cos 4t \equiv A\sin 4t\cos\alpha + A\cos 4t\sin\alpha$

Equating parts,

$2\sqrt{3}\sin 4t = A\sin 4t\cos\alpha$ | $6\cos 4t = A\cos 4t\sin\alpha$.

$2\sqrt{3} = A\cos\alpha$ (1) | $6 = A\sin\alpha$ (2)

Squaring & adding (1) & (2) $A^2(\cos^2\alpha + \sin^2\alpha) = (2\sqrt{3})^2 + 6^2$

$A = \sqrt{48} = 4\sqrt{3}$.

Sub. $A = 4\sqrt{3}$ in (1): $2\sqrt{3} = 4\sqrt{3}\cos\alpha$

$\frac{1}{2} = \cos\alpha$.

Sub. $A = 4\sqrt{3}$ in (2):

$6 = 4\sqrt{3}\sin\alpha$.

$\frac{\sqrt{3}}{2} = \sin\alpha$

$\alpha = \frac{\pi}{3}$.

$\therefore x = 4\sqrt{3}\sin\left(4t + \frac{\pi}{3}\right)$

Period $= \frac{2\pi}{4} = \frac{\pi}{2}$. Amplitude $= 4\sqrt{3}$.

Marker's Comments

Mostly done well

Solutions: p. 10

Q.(13)'b) $4v^2 = 8 - x^2 - 2x$.

$$\frac{d}{dx}(4v^2) = -2x - 2$$

$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{1}{4}(x+1)$$

As $\ddot{x} = -n^2(x-C)$, $n = \frac{1}{2}$, $C = -1$,

particle is moving with SHM.

Period of motion $= \frac{2\pi}{\left(\frac{1}{2}\right)} = 4\pi$ seconds.

Amplitude: If $v = 0$, $4v^2 = 8 - x^2 - 2x = 0$

$$x^2 + 2x - 8 = 0$$

$$(x+4)(x-2) = 0$$

Particle oscillating between $x = -4$ & $x = 2$.

Centre of motion is $x = -1$. Amplitude $= 3$. Period $= 4\pi$ seconds.

Note: Can also do: $4v^2 = -(x^2 + 2x - 8)$

$$4v^2 = 9 - (x+1)^2$$

$$v^2 = \frac{1}{4}\left[9 - (x+1)^2\right].$$

Marker's Comments

Instead of $\ddot{x} = -\frac{1}{4}(x+1)$ some students are showing $v^2 = \ldots$

They lost 1 marks

Solutions: p.11

Marker's Comments

Q.(13)(c)(i) Assume $\sqrt{7}$ = rational

i.e. $\sqrt{7} = \frac{p}{q}$, $p, q \in \mathbb{Z}$

& p, q have no common factors.

$\therefore \frac{p^2}{q^2} = 7$.

$p^2 = 7q^2$ (1)

As p, q have no common factors, 7 is a factor of p^2

$\Rightarrow$ 7 is a factor of p

$\therefore p = 7k$

$p^2 = (7k)^2 = 49k^2$ (2)

done well

Sub. (2) in (1): $49k^2 = 7q^2$

$7k^2 = q^2$.

7 is a factor of $q^2 \Rightarrow$ 7 is a factor of q.

But p & q have no common factors.

There is a contradiction $\Rightarrow \sqrt{7}$ is irrational.

(ii) Contrapositive: The square of a rational number is rational.

Let rational no $x = \frac{p}{q}$, $p, q \in \mathbb{Z}$.

$x^2 = \frac{p^2}{q^2}$, which is also rational.

$\therefore$ The square root of an irrational number must be irrational [by contraposition].

have difficulty in writing the contrapositive statement.

(iii) $(\sqrt{2}+\sqrt{7})^2$

$= 2 + 7 + 2\sqrt{14}$

$= 9 + 2\sqrt{14}$

which is irrational, as $\sqrt{14}$ is irrational & the sum of a rational & irrational no. is irrational

Many are not using 'hence' So lost 1 mark

$\therefore$ As $(\sqrt{2}+\sqrt{7})^2$ is irrational,

$\sqrt{2}+\sqrt{7}$ is irrational by (ii).

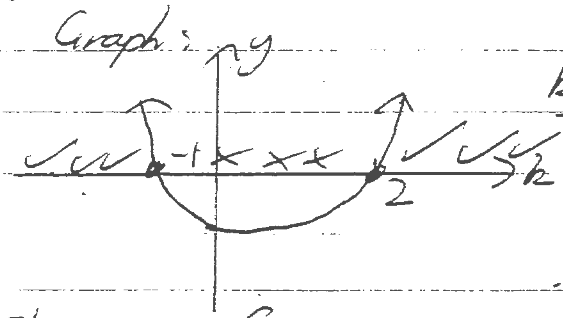
Solutions: p. 12

Q.(14)(a)(i) Solving $k^2 - k - 2 \geq 0$

$(k-2)(k+1) \geq 0$

done well.

Graph:



$k \geq 2$ or $k \leq -1$.

Markers' comments

(ii) Step 1: Show true for $n = 2$

LHS	RHS
$= 3^2$	$= 2 \times 2^2 - 3$
$= 9$	$= 5$

LHS > RHS True for $n=2$.

done well

Step 2: Assume true for $n = k$

i.e. $3^k > 2k^2 - 3,\ k \geq 2.$

Step 3: Prove true for $n = k+1$

i.e. RTP: $3^{k+1} > 2(k+1)^2 - 3$

or $3^{k+1} > 2k^2 + 4k - 1,\ k \geq 2.$ [See note below ⊛]

LHS

3^{k+1}

$= 3 \times 3^k$

$> 3[2k^2 - 3]$ [by assumption]

$= 6k^2 - 9$

$> 2k^2 + 4k - 1$ [as $6k^2 - 9 - (2k^2 + 4k - 1) = 4k^2 - 4k - 8$
$= 4(k^2 - k - 2) > 0$
where $k \geq 2$
from (i)].

$3^{k+1} > 6k^2 - 9 > 2k^2 + 4k - 1$

$3^{k+1} > 2k^2 + 4k - 1.$

True for $n = k+1$. ∴ If true for $n=k$ it will be true for $n=k+1$. Hence as it is true for $n=2$ it will be true for all $n \in Z \geq 2$ by the principle of mathematical induction.

⊛ Note: Can also do RTP $3^{k+1} - (2k^2 + 4k - 1) > 0$

Steps: $3^{k+1} - (2k^2 + 4k - 1)$

$> 3(2k^2 - 3) - (2k^2 + 4k - 1)$

$= \underline{4k^2 - 4k - 8 = 4(k^2 - k - 2) > 0 \ \forall\ k \geq 2 \text{ from (i)}}$

Solutions: p.13

Marker's comments

Q.(14)(b)(i) If $a, b \in \mathbb{R}$, $a-b \in \mathbb{R}$

$$(a-b)^2 \geq 0$$
$$a^2+b^2-2ab \geq 0$$
$$a^2+b^2 \geq 2ab$$

done well

(ii) If a, b, c real

$a^2+b^2 \geq 2ab$ (1) from (i)

$a^2+c^2 \geq 2ac$ (2)

$b^2+c^2 \geq 2bc$ (3)

done well

(1) + (2) + (3) $2(a^2+b^2+c^2) \geq 2(ab+ac+bc)$

$$a^2+b^2+c^2 \geq ab+ac+bc$$

(iii) $(a+b+c)^2$

$= a^2+b^2+c^2+2(ab+ac+bc)$

$\leq a^2+b^2+c^2+2(a^2+b^2+c^2)$ [using (ii)]

$= 3(a^2+b^2+c^2)$

$= 3 \times 27$

Many got wrong.

$= 81.$

$\therefore (a+b+c)^2 \leq 81$

$-9 \leq a+b+c \leq 9.$

(iv) $(x-1)^2+(y+2)^2+(z+3)^2 = 27.$

Centre $= (1, -2, -3)$ $r = 3\sqrt{3}$ units.

done well

(v) If $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ is a point on the surface of the sphere

then $(x_1-1)^2+(y_1+2)^2+(z_1+3)^2 = 27.$

$\therefore -9 \leq (x_1-1)+(y_1+2)+(z_1+3) \leq 9$ [by (iii)].

$\quad -4 \qquad\qquad\qquad -4 \quad -4$

$-13 \leq x_1 + y_1 + z_1 \leq 5.$

as required.

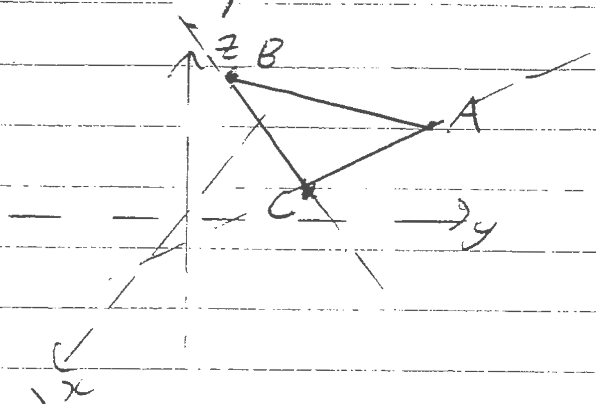
writing a general statement like

$-13 \leq x+y+z \leq 5$

lost mark.

Solutions: p. 14

Q. (14)(a)(i)



$$\begin{pmatrix}8\\14\\18\end{pmatrix}+\lambda\begin{pmatrix}3\\4\\0\end{pmatrix}=\begin{pmatrix}-16\\-18\\28\end{pmatrix}+\mu\begin{pmatrix}3\\4\\-5\end{pmatrix}$$

done well.

Equating

x: $8+3\lambda = -16+3\mu \Rightarrow \lambda-\mu=-8$ (1)

y: $14+4\lambda = -18+4\mu \Rightarrow \lambda-\mu=-8$ (2) } SAME!

z: $18 = 28-5\mu \Rightarrow 5\mu=10 \Rightarrow \mu=2$.

Sub. $\mu=2$ in (1): $\lambda-2=-8 \Rightarrow \lambda=-6$.

Check: $\begin{pmatrix}8\\14\\18\end{pmatrix}-6\begin{pmatrix}3\\4\\0\end{pmatrix}=\begin{pmatrix}-10\\-10\\18\end{pmatrix}$. & $\begin{pmatrix}-16\\-18\\28\end{pmatrix}+2\begin{pmatrix}3\\4\\-5\end{pmatrix}=\begin{pmatrix}-10\\-10\\18\end{pmatrix}$.

$B=\begin{pmatrix}-10\\-10\\18\end{pmatrix}$

Marker's Comments

(ii) $\cos\angle ABC = \dfrac{\begin{pmatrix}3\\4\\0\end{pmatrix}\cdot\begin{pmatrix}3\\4\\-5\end{pmatrix}}{\sqrt{25}\times\sqrt{50}}$

$= \dfrac{25}{25\sqrt{2}}$

$= \dfrac{1}{\sqrt{2}}$.

$\angle ABC = 45^\circ$.

Few got wrong.

Solutions: p.15 Marker's comments

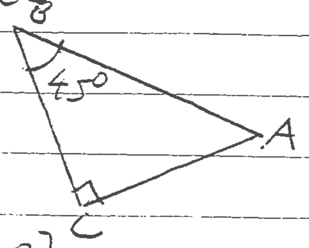
Q.(14)(c) [continued].

(iii) Direction AC · Direction BC

$= \begin{pmatrix} -3 \\ -4 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$

$= 0.$

$\therefore AC \perp BC.$



Hence $\angle CAB = 45^\circ$ ($\angle$ sum $\triangle ABC$).

$\triangle ABC$ is isosceles [2 $\angle$'s =]. done well.

(iv) Vector $\vec{AB} = \begin{pmatrix} -18 \\ -24 \\ 0 \end{pmatrix}$. Length $AB = \sqrt{18^2 + 24^2} = 30$ units.

Direction $\vec{AC} = \begin{pmatrix} -3 \\ -4 \\ -5 \end{pmatrix}$.

$\therefore$ Length $\vec{AC} = \left| Proj_{\vec{AC}} \vec{AC} \right| = \dfrac{\begin{pmatrix} -18 \\ -24 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -4 \\ -5 \end{pmatrix}}{\sqrt{3^2+4^2+5^2}} = \dfrac{150}{5\sqrt{2}} = 15\sqrt{2}$ units.

By Area $= \frac{1}{2} \times |\vec{AB}| \times |\vec{AC}| \times \sin \angle BAC$

Area $= \frac{1}{2} \times 30 \times 15\sqrt{2} \times \sin 45^\circ \left(\frac{1}{\sqrt{2}}\right)$

$= 225u^2.$

lots of different answers. Some find it hard. Those who wrote the correct formula were given 1 mark.

Note: We could also find C [intersection of BC & AC]

& $C = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$. Then $\vec{AC} = \begin{pmatrix} -9 \\ -12 \\ -15 \end{pmatrix}$ & $|\vec{AC}| = \sqrt{450} = 15\sqrt{2}$.

& $\vec{BC} = \begin{pmatrix} 9 \\ 12 \\ -15 \end{pmatrix}$ & $|\vec{BC}| = 15\sqrt{2}$.

Then $A = \frac{1}{2} \times \vec{BC} \times \vec{AC} = \frac{1}{2} \times 15\sqrt{2} \times 15\sqrt{2} = 225\,u^2.$

Solutions: p.16

Q.(15)(a)(i)

$\rightarrow$ BALL. $\quad \longleftarrow$ 0·5N Headwind.

$\longleftarrow$ 0·05v air resistance.

$$F = ma = 0\cdot1a = -0\cdot5 - 0\cdot05v.$$
$$a = -5 - 0\cdot5v.$$

(ii) $\ddot{x} = \dfrac{dv}{dt} = -5 - 0\cdot5v$

$$\frac{dt}{dv} = \frac{-1}{5+0\cdot5v}$$
$$= \frac{-2}{10+v}$$
$$t = \int_{20}^{v} \frac{-2}{10+v}\,dv$$
$$= \Big[-2\ln(10+v)\Big]_{20}^{v}$$
$$= -2\left[\ln(10+v) - \ln(10+20)\right]$$
$$t = -2\ln\left(\frac{10+v}{30}\right)$$

Ball STOPS when $v=0 \Rightarrow t = -2\ln\left(\dfrac{10}{30}\right)$
$$= 2\ln 3$$
$$\doteqdot 2\cdot197\ldots \text{ seconds}.$$

[Note: you also do the INDEFINITE integral with $t = -2\ln(10+v)+C$, $v=20$ when $t=0$. etc.

Marker's Comments

Marker's Comments

ai, done very well.

aii, however done very well however Students missed doing the second part of the question however marks paid if 't' was found in aiii,

Solutions: p.17

Q.(15)(a)(iii) $t = -2\ln\left(\frac{10+v}{30}\right)$

$$e^{-0.5t} = \frac{10+v}{30}$$

done very well

$$30e^{-0.5t} - 10 = v.$$

$$x = \int_0^t 30e^{-0.5t} - 10 \, dt$$

$$= \left[-60e^{-0.5t} - 10t\right]_0^t$$

$$= -60e^{-0.5t} - 10t + 60 \text{ as required.}$$

When $t = 2\ln 3$,

$$x = -60e^{\ln(\frac{1}{3})} - 10 \times 2\ln 3 + 60$$

$$= 18.0277\ldots m.$$

Ball stops after 18.03m (2DP).

Marker's Comments

Some students lost marks due to Calculation Error and some students skipped the second part of the Question entirely.

Solutions: p.18

Q.(15)(b)(i) $\ddot{x} = 10 - \frac{v^2}{40}$.

$\downarrow 10$

$\frac{v^2}{40} \uparrow$

Marker's Comments

Very well done.

Terminal v is when $a=0$

$$10 - \frac{v^2}{40} = 0$$

$$\underline{v = 20 m/s.}$$

(ii) $\ddot{x} = v \cdot \frac{dv}{dx} = 10 - \frac{v^2}{40} = \frac{400 - v^2}{40}$

$$\frac{dv}{dx} = \frac{400 - v^2}{40v}.$$

$$\frac{dx}{dv} = \frac{40v}{400 - v^2}$$

$$x = -20 \int_0^V \frac{-2v}{400 - v^2}\, dv$$

as initial $v=0$ & goes to x when $v=V$.

$$x = -20 \left[\ln(400 - v^2)\right]_0^V$$

$$\underline{x = -20 \ln\left[\frac{400 - v^2}{400}\right] \text{ as required.}}$$

Parcel hits ground: $x = 50m$.

$$50 = -20 \ln\left(\frac{400 - v^2}{400}\right)$$

$$-2.5 = \ln\left(\frac{400 - v^2}{400}\right)$$

$$e^{-2.5} = \frac{400 - v^2}{400}$$

$$v^2 = 400(1 - e^{-2.5})$$

$$\underline{v = 19.16 m/s. \ [2DP].}$$

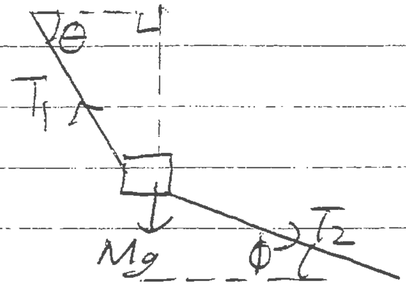
Marker's Comments

Very well done

Solutions: p. 19

Q. (15)(c)(i)

Marker's Comments



Very well done

Resolving horizontally:

$T_1 \cos\theta = T_2 \cos\phi$. (1)

Resolving vertically: Note: Raised at CONSTANT rate, so acceleration = 0 & forces in balance:

$\therefore T_1 \sin\theta = T_2 \sin\phi + Mg$ (2)

(2) $\div T_1 \cos\theta$ & noting that $T_1 \cos\theta = T_2 \cos\phi$

$$\frac{T_1 \sin\theta}{T_1 \cos\theta} = \frac{T_2 \sin\phi}{T_2 \cos\phi} + \frac{Mg}{T_2 \cos\phi}$$

$$\tan\theta = \tan\phi + \frac{Mg}{T_2 \cos\theta}$$ as required.

(ii) Note from (1) $\tan\theta > \tan\phi$.

$$\therefore \frac{L}{3} > \frac{12-L}{9} \quad \times 9$$

$$3L > 9 - L$$

$$\therefore 4L > 12$$

$$\underline{L > 3.}$$

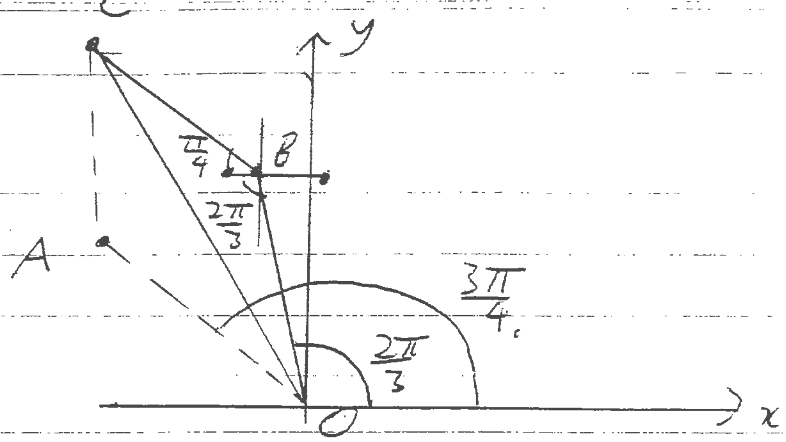
Students struggled with reasoning.

So the machine can't be raised more than $12 - 3 = 9$m above the floor.

Marker's Comments

Solutions: p. 20

Q. (16)(a)(i)



Note: $\angle OBC = \frac{2\pi}{3} + \frac{\pi}{4}$.

Using the cosine rule on $\triangle OBC$,

$$(OC)^2 = |z+w|^2 = 1^2 + 1^2 - 2\times 1\times 1 \times \cos\left(\frac{2\pi}{3}+\frac{\pi}{4}\right)$$

Alternative: Could also have done $|z+w|^2$ directly $= \left(\frac{-\sqrt{2}-1}{2}\right)^2 + \left(\frac{\sqrt{2}+\sqrt{3}}{2}\right)^2$ etc.

$$= 2 - 2\left[\cos\frac{2\pi}{3}\cos\frac{\pi}{4} - \sin\frac{2\pi}{3}\sin\frac{\pi}{4}\right]$$

$$= 2 - 2\left[-\frac{1}{2}\times\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2}\right]$$

$$= 2 + \frac{\sqrt{2}+\sqrt{6}}{2}$$

Could also use $|z+w|^2 = (z+w)(\bar{z}+\bar{w})$

Note: The new fx8200 calculators CAN get $\cos\frac{\pi}{12}$ directly & markers would have to pay it.

$$|z+w|^2 = \frac{4+\sqrt{2}+\sqrt{6}}{2}$$

(ii) Using $\triangle OBC$: $\angle COB = \frac{1}{2}\left[\pi - \left(\frac{2\pi}{3}+\frac{\pi}{4}\right)\right]$ [$\angle$'s opposite = sides in isosceles $\triangle COB$]

$$= \frac{\pi}{24}.$$

or $\angle COB = \frac{1}{2}\left(\frac{3\pi}{4} - \frac{2\pi}{3}\right)$ [Diagonals bisect $\angle$'s in rhombus OACB].

$$= \frac{\pi}{24}$$

Marker's Comments

GLARING problem: You had to say diagonals of a RHOMBUS here!

Diagonals of a parallelogram DON'T bisect each other.

$\text{Arg}(z+w) \neq \frac{\text{Arg}(z)+\text{Arg}(w)}{2}$ UNLESS $|z| = |w|$ FOR THIS REASON.

Solutions: p. 21

Marker's Comments

Q.(16)(a)(iii) $\frac{z+w}{w}$

$= 1 + \frac{z}{w}$

$= 1 + \frac{-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}}{-\frac{1}{2} + i\frac{\sqrt{3}}{2}}$

$= 1 + \frac{(-\sqrt{2}+i\sqrt{2})}{-1+i\sqrt{3}} \times \frac{(-1-i\sqrt{3})}{(-1-i\sqrt{3})}$

$= 1 + \frac{\sqrt{2}+i\sqrt{6}-i\sqrt{2}+\sqrt{6}}{4}$

$= \frac{4+(\sqrt{2}+\sqrt{6})}{4} + \frac{i(\sqrt{6}-\sqrt{2})}{4}$

as required.

Students could also do $\frac{z+w}{w}$ directly, although this was messier & resulted in silly mistakes.

Also: some students with fx-8200 calculator did $1 + \frac{z}{w} = 1 + \text{cis}\frac{\pi}{12}$ & as they can get $\text{cis}\frac{\pi}{12}$ on fx-8200 I had to pay it.

(iv) Hence as $\left|\frac{z+w}{w}\right| = \frac{|z+w|}{1} = \sqrt{\frac{4+\sqrt{2}+\sqrt{6}}{2}}$ [from (i)]

& $\text{Arg}\left(\frac{z+w}{w}\right) = \frac{\pi}{24} = \angle COB$ (from (ii)).

$\cos\frac{\pi}{24} = \frac{4+\sqrt{2}+\sqrt{6}}{4} \div \frac{\sqrt{4+\sqrt{2}+\sqrt{6}}}{\sqrt{2}}$

$= \frac{4+\sqrt{2}+\sqrt{6}}{4} \times \frac{\sqrt{2}}{\sqrt{4+\sqrt{2}+\sqrt{6}}}$

$= \frac{\sqrt{2}\sqrt{4+\sqrt{2}+\sqrt{6}}}{4}$.

Students had to show this step for the mark.

*

Solutions: p. 22

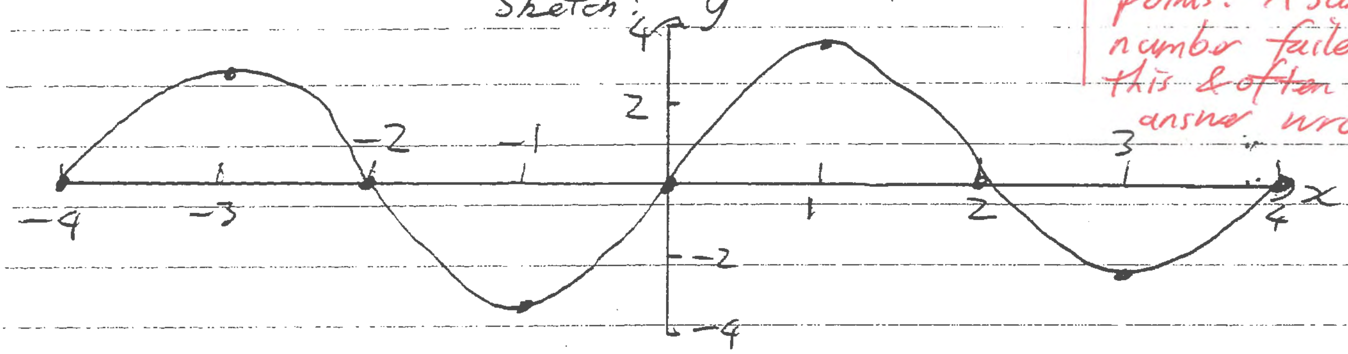
Q.(16)(b)(i)

x	-4	-3	-2	-1	0
$y=\sqrt{16-x^2}\sin\left(\frac{\pi}{2}x\right)$	0	$\sqrt{7}\sin\left(-\frac{3\pi}{2}\right)$ $=\sqrt{7}$	$\sqrt{12}\sin(-\pi)$ $=-0$	$\sqrt{15}\sin\left(-\frac{\pi}{2}\right)$ $=-\sqrt{15}$	0

x	1	2	3	4
y	$\sqrt{15}\sin\frac{\pi}{2}$ $=\sqrt{15}$	$\sqrt{12}\sin\pi$ $=0$	$\sqrt{7}\sin\frac{3\pi}{2}$ $=-\sqrt{7}$	0

This was a simple question where all students had to do was fill out a table & plot points. A surprising number failed to do this & often got the answer wrong.

Sketch:



(ii) Table:

x	$-\sqrt{7}$	0	$\sqrt{15}$	0	0	0
y	3	0	1	2	4	-4
z	0	4	0	$-\sqrt{12}$	0	0

→ Spiral is following y axis. * BIG OFF.

Also y has all whole values.

∴ If we let $y=t$.

Looking above we can see

$x=\sqrt{15}$ when $t=1$ & $x=-\sqrt{7}$ when $y=3$

$\therefore x=\sqrt{16-t^2}\,\sin\frac{\pi}{2}t$.

$$\text{As } x^2+y^2+z^2 = 16 \quad \text{(as spiral is on sphere).}$$

$$(16-t^2)\sin^2\tfrac{\pi}{2}t + t^2 + z^2 = 16$$

$$z^2 = (16-t^2)-(16-t^2)\sin^2\left(\tfrac{\pi}{2}t\right)$$

$$= (16-t^2)\left[1-\sin^2\left(\tfrac{\pi}{2}t\right)\right]$$

$$z = \sqrt{16-t^2}\,\cos\left(\tfrac{\pi}{2}t\right)$$

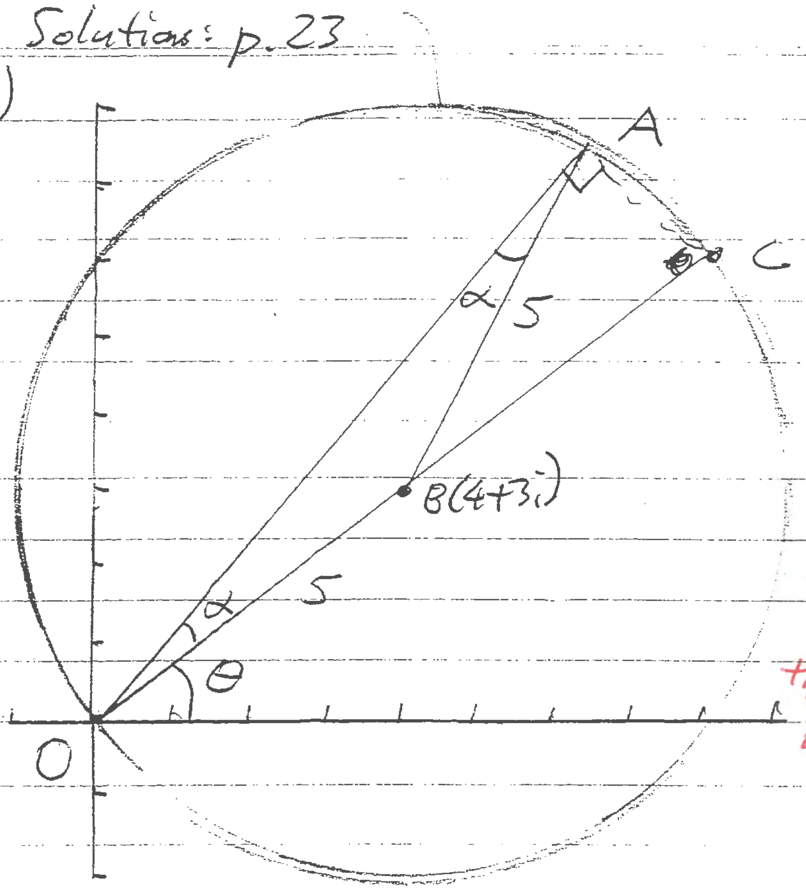
∴ Parametric equation of spiral is

$$\underline{\left(\sqrt{16-t^2}\,\sin\tfrac{\pi}{2}t,\ t,\ \sqrt{16-t^2}\,\cos\tfrac{\pi}{2}t\right)}.$$

The vast majority of students could NOT see the connection between (i) & (ii). If they had USED A TABLE [& the students who did often saw at least part of the solution] they would have had more success.

Solutions: p.23

Q.(16)(c)



Many people had the right idea here, but they needed to DRAW ∴ A DIAGRAM.

(i) $OB = BA = 5$ units.

$\angle OAB = \alpha$ [$\angle$ sum isosceles $\triangle OAB$].

$\angle OBA = 180^\circ - 2\alpha$.

By cosine rule on $\triangle OBA$, $|z|^2 = (OA)^2 = 5^2 + 5^2 - 2\times5\times5\times\cos(180^\circ - 2\alpha)$

$$= 50 + 50\cos 2\alpha.$$

$$= 50 + 50(2\cos^2\alpha - 1)$$

$$|z|^2 = 100\cos^2\alpha$$

$$\therefore |z| = 10\cos\alpha.$$

(Or) Label C above.

$\angle OAC = 90^\circ$ [$\angle$ in semicircle].

Using SOHCAHTOA on $\triangle OAC$: $\cos\alpha = \frac{OA}{10}$

$$10\cos\alpha = OA$$

$$|z| = 10\cos\alpha$$

Solutions: p.24

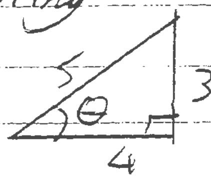
$$w = \frac{1}{z} \quad \therefore |w| = \frac{1}{10\cos\alpha} \quad \& \ \operatorname{Arg} w = -(\alpha + \theta).$$

$$\therefore \quad w = \frac{1}{10\cos\alpha}\left[\cos[-(\alpha+\theta)] + i\sin[-(\alpha+\theta)]\right]$$

$$= \frac{1}{10\cos\alpha}\left[\cos(\alpha+\theta) - i\sin(\alpha+\theta)\right]$$

Students were often sloppy here & didn't write cos even, sin odd etc. & lost marks.

Noting



$\cos\theta = \frac{4}{5}$ & $\sin\theta = \frac{3}{5}$,

Note: they could also use $\frac{1}{z} = \cdots \ \frac{1}{10\cos\alpha}\bar{z}$.

$$w = \frac{1}{10\cos\alpha}\left[(\cos\alpha\cos\theta - \sin\alpha\sin\theta) - i(\sin\alpha\cos\theta + \cos\alpha\sin\theta)\right]$$

$$= \frac{1}{10\cos\alpha}\left[\left(\frac{4}{5}\cos\alpha - \frac{3}{5}\sin\alpha\right) - i\left(\frac{4}{5}\sin\alpha + \frac{3}{5}\cos\alpha\right)\right]$$

as required.

(iii) For w, $x = \dfrac{\frac{4}{5}\cos\alpha - \frac{3}{5}\sin\alpha}{10\cos\alpha}$ $\quad$ (*) $\qquad y = \dfrac{-\frac{4}{5}\sin\alpha - \frac{3}{5}\cos\alpha}{10\cos\alpha}$

$x = \frac{4}{50} - \frac{3}{50}\tan\alpha. \qquad y = -\frac{4}{50}\tan\alpha - \frac{3}{50}.$

$200x = 16 - 12\tan\alpha. \qquad 150y = -12\tan\alpha - 9.$

$$200x - 150y = 16 - 12\tan\alpha + 12\tan\alpha + 9$$

$$200x - 150y = 25.$$

$$\underline{8x - 6y = 1.} \quad \text{or } \underline{y = \frac{4}{3}x - \frac{1}{6}.}$$

Note: You do NOT need to find the domain or range of w] as noted in Q3.

END OF SOLUTIONS!!!

Many students got to (*) but could not see to convert to $\tan\alpha$.

Most students who got to $\tan\alpha$ realised they could get x & y in terms of $\tan\alpha$.